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Shape reconstruction of an inverse Stokes problem☆

Wen-Jing Yan*, Yi-Chen Ma

School of Science, Xi'an Jiaotong University, P.O. Box 1844, Xi'an, Shaanxi 710049, PR China

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Abstract

This paper deals with the shape reconstruction of a viscous incompressible fluid driven by the Stokes flow. For the approximate solution of the ill-posed and nonlinear problem we propose a regularized Newton method. A theoretical foundation for the Newton method is given by establishing the differentiability of the initial boundary value problem with respect to the interior boundary curve in the sense of a domain derivative. The numerical examples show that our theory is useful for practical purpose and the proposed algorithm is feasible.

Keywords: Domain derivative; Shape reconstruction; Stokes equations; Inverse problem

1. Introduction

This paper is concerned with the problem of the shape reconstruction of two-dimensional flows governed by Stokes equations. This problem is a basic tool in the design of many industrial devices such as aircraft wings, automobile shapes, boats, and so on.

Early works concerning with domain derivative have been addressed in [8,6,2,3]. Kirsch and Hettlich solved the inverse obstacle scattering problem for sound soft and sound hard obstacles, and, in [2,3], the three authors deal with the inverse boundary problem for the time-dependent heat equation only in the case of perfectly conducting and insulating inclusions. In [10], we solve a shape reconstruction problem for heat conduction with mixed condition, and, in [11], we derive the expressions of domain derivative for the steady Navier–Stokes equations. To the author's knowledge, there are few studies considering the shape reconstruction of Stokes flow. Our concerns in this article are establishing the domain derivative of the Stokes equations in a multiple bounded domain, and deriving an efficient numerical approach for the solution of the two-dimensional realizations of such problem.

As we all know, the divergence free condition coming from the fact that the fluid has a homogeneous density and evolves as an incompressible flow is very difficult to impose on the mathematical and numerical point of view. We use Piola transformation to bypass the divergence free condition for the flow.

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* Corresponding author.
E-mail addresses: sansery@126.com (W.-J. Yan), ycma@mail.xjtu.edu.cn (Y.-C. Ma).

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The plan of the paper is as follows. In the remainder of the section we establish the notation that will be used throughout the work. Section 2 is devoted to introduce Piola transformation for divergence free condition and we establish the differentiability of the solution with respect to the boundary. The techniques we used are similar to those used in [10,11]. The third section is devoted to regularized Newton schemes applied to the numerical inverse problem. The results of several numerical experiments show that the iterative algorithm gives good reconstruction, and indicate the feasibility of this method.

Throughout the paper we will use the standard notation for Sobolev spaces (see [1]). Specially $H^r(\Omega)$, where r is an integer greater than zero, will denote the Sobolev space of real-valued functions with square integrable derivatives of order up to r equipped with the usual norm which we denote $\|\cdot\|_r$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$, and the standard L^2 inner product by $(\cdot,\cdot)$. Also $\mathbf{H^r(\Omega)}$ will denote the space of vector-valued functions each of whose n components belong to $H^r(\Omega)$. We introduce the space

$$\mathbf{V}(\Omega) := \{\boldsymbol{v} \in \mathbf{H^1}(\Omega), \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v}|_{\partial\Omega} = 0\}.$$

2. Domain derivative

We assume that Ω_1 and Ω_2 are two simply connected bounded domains of class C^2 in $R^{\mathbb{N}}$ ($\mathbb{N} = 2$ or 3) such that $\bar{\Omega}_2 \subset \Omega_1$. The boundaries of Ω_1 and Ω_2 are denoted by Γ_1 and Γ_2, respectively. Further, we denote $\Omega := \Omega_1 \backslash \bar{\Omega}_2$. Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ be a given vector function in Ω. We seek a vector function $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ representing the velocity of the fluid, and a scalar function p representing the pressure, which are defined in Ω and satisfy the following equations and boundary condition (v is the coefficient of kinematic viscosity):

$$\begin{cases} -v\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma_1, \\ \boldsymbol{u} = 0 & \text{on } \Gamma_2. \end{cases} \tag{2.1}$$

Taking the scalar product of (2.1) with a function $\boldsymbol{v} \in \mathbf{V}(\Omega)$ we obtain

$$\begin{cases} \text{seek } \boldsymbol{u} \in \mathbf{V}(\Omega) \text{ such that:} \\ a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \end{cases} \tag{2.2}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = v \sum_{i,j=1}^{n} \int_\Omega (D_i u_j)(D_i v_j)\,\mathrm{d}x.$$

Continuity of the forms $a(\cdot,\cdot)$ can be demonstrated. Obviously, the form $a(\cdot,\cdot)$ is H^1 coercive according to the classical theory of elliptic system. These conditions guarantee the existence and uniqueness of a solution $\boldsymbol{u}$ (see [9,5]).

Let a perturbation of the interior boundary Γ_2 be specified by

$$\Gamma_2^h = \{\boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}), \boldsymbol{x} \in \Gamma_2\},$$

which is a C^2 boundary of a perturbed domain Ω_h, if the vector field $\boldsymbol{h} \in C^2(\Gamma_2)$ is sufficiently small. We choose an extension of $\boldsymbol{h} \in C^2(\Omega)$ with $\|\boldsymbol{h}\|_{C^2(\Omega)} \leqslant c\|\boldsymbol{h}\|_{C^2(\Gamma_2)}$, $c > 0$, which vanishes in the exterior of a neighborhood of Γ_2, and define the diffeomorphism $\varphi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{h}(\mathbf{x})$ in Ω. If the inverse function of φ is denoted by ψ, J_φ and J_ψ are the Jacobian matrices.

We introduce the Piola transformation:

$$P : \boldsymbol{w} \to (\det(J_\varphi)^{-1} J_\varphi \tilde{\boldsymbol{w}}) \circ \psi,$$

where $\boldsymbol{w} \in \mathbf{V}(\Omega_h)$ and $\tilde{\boldsymbol{w}} \in \mathbf{V}(\Omega)$. It still satisfies the condition of divergence free by the transformation. q and q_h represent the test functions in $\mathbf{V}(\Omega)$ and $\mathbf{V}(\Omega_h)$ respectively.

$$\begin{aligned}\int_{\Omega_h} \operatorname{div} \boldsymbol{w} \cdot q_h \, \mathrm{d}x_h &= -\int_{\Omega_h} \boldsymbol{w} \cdot \nabla q_h \, \mathrm{d}x_h + \int_{\partial\Omega_h} \boldsymbol{w} \cdot q_h \cdot \boldsymbol{n} \, \mathrm{d}s_h \\ &= -\int_{\Omega_h} (\det(J_\varphi)^{-1} J_\varphi \tilde{\boldsymbol{w}} \circ \psi) \cdot \nabla(q_h \circ \psi) \, \mathrm{d}x_h \\ &= -\int_{\Omega_h} (\det(J_\varphi)^{-1} J_\varphi \tilde{\boldsymbol{w}} \circ \psi) \cdot (\nabla q_h J_\varphi^{-1} \circ \psi) \, \mathrm{d}x_h \\ &= \int_{\Omega} \tilde{\boldsymbol{w}} \cdot \nabla q \, \mathrm{d}x = \int_{\Omega} \operatorname{div} \tilde{\boldsymbol{w}} \cdot q \, \mathrm{d}x.\end{aligned}$$

Let $\boldsymbol{u}_h \in \mathbf{V}(\Omega_h)$ be the solution of corresponding boundary value problem, i.e., satisfying the variational equation

$$v \int_{\Omega_h} \nabla \boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h \, \mathrm{d}x_h = \int_{\Omega_h} \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}x_h \quad \forall \boldsymbol{v}_h \in \mathbf{V}(\Omega_h). \tag{2.3}$$

Changing the variables by the Piola transformation leads to

$$v \int_{\Omega} \nabla(B\tilde{\boldsymbol{u}}) \cdot \nabla(B\boldsymbol{v}) A \, \mathrm{d}x = \int_{\Omega} \tilde{\boldsymbol{f}} \cdot \boldsymbol{v} J_\varphi \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \tag{2.4}$$

where $B := \det(J_\varphi)^{-1} J_\varphi$, $A := J_\varphi^{-1}(J_\varphi^{-1})^{\mathrm{T}} \det(J_\varphi)$, and $\tilde{\boldsymbol{f}} = \boldsymbol{f}_h \circ \psi$.

Denote the Jacobian of h by J_h. From $J_\varphi = I + J_h$ and $J_\psi = J_\varphi^{-1} \circ \psi = I - J_h + \mathrm{O}(\|\boldsymbol{h}\|^2_{C^2(\Omega)})$, the following estimates hold:

$$\|J_\varphi^{-1}(J_\varphi^{-1})^{\mathrm{T}} \det(J_\varphi) - I + J_h + J_h^{\mathrm{T}} - \operatorname{div} \boldsymbol{h} \cdot I\|_\infty = \mathrm{O}(\|\boldsymbol{h}\|^2_{C^2(\Omega)}), \tag{2.5}$$

$$\|\tilde{\boldsymbol{f}} \cdot J_\varphi - \boldsymbol{f} - \boldsymbol{f} \cdot \operatorname{div} \boldsymbol{h} - \boldsymbol{h} \cdot \nabla \boldsymbol{f}\|_\infty = \mathrm{O}(\|\boldsymbol{h}\|^2_{C^2(\Omega)}), \tag{2.6}$$

and

$$\|J_\varphi^{-1} \det(J_\varphi) - I + J_{\boldsymbol{h}} - \operatorname{div} \boldsymbol{h} \cdot I\|_\infty = \mathrm{O}(\|\boldsymbol{h}\|^2_{C^2(\Omega)}). \tag{2.7}$$

Lemma 2.1 (*Hettlich [6,7]*). *If $u_i, v_i \in H_0^1(\Omega)$, $i = 1, \ldots, N$, then the following identity holds*:

$$\nabla u_i (J_h + J_h^{\mathrm{T}} - \operatorname{div} \boldsymbol{h} \cdot I) \nabla v_i = \operatorname{div}[\boldsymbol{b}_i] - (\boldsymbol{h} \cdot \nabla u_i)\Delta v_i - (\boldsymbol{h} \cdot \nabla v_i)\Delta u_i, \tag{2.8}$$

where $\boldsymbol{b}_i = (\boldsymbol{h} \cdot \nabla u_i)\nabla v_i + (\boldsymbol{h} \cdot \nabla v_i)\nabla u_i - (\nabla u_i \cdot \nabla v_i)\boldsymbol{h}$.

Lemma 2.2 (*Delfour and Zolésio [4]*). *Let $w \in C^2(\Gamma)$, $a \in C^1(\Gamma)$ be two scalar functions, and a vector field $\boldsymbol{v} \in C^1(\Gamma)^N$. The following decompositions hold*:

$$\nabla w = \nabla_\tau w + \partial_{\boldsymbol{n}} w \boldsymbol{n}, \tag{2.9}$$

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} + v_\tau, \quad v_\tau = \boldsymbol{n} \wedge (\boldsymbol{v} \wedge \boldsymbol{n}). \tag{2.10}$$

Moreover, we can prove the following important result which is the main theoretical result of the paper.

Theorem 2.1. *Let $\boldsymbol{f}$ be sufficiently smooth, and $\boldsymbol{u} \in \mathbf{V}(\Omega)$ denote the solution of* (2.1). *$\boldsymbol{u}_h$ is defined on the perturbed domain Ω_h, and $\tilde{\boldsymbol{u}}$ is defined in* (2.4). *Then $\boldsymbol{u}$ is differentiable at Γ_2 in the sense that there exists $\boldsymbol{u}^*$ depending on $\boldsymbol{h}$*

such that

$$\lim_{\boldsymbol{h}\to 0}\frac{1}{\|\boldsymbol{h}\|_{C^2}}|\tilde{\boldsymbol{u}}-\boldsymbol{u}-\boldsymbol{u}^*|_1=0. \tag{2.11}$$

Furthermore $\boldsymbol{u}^*:=\boldsymbol{u}'+\boldsymbol{h}\cdot\nabla\boldsymbol{u}$, *where the domain derivative* $\boldsymbol{u}'$ *is defined by the solution of the boundary value problem*

$$\begin{cases} -\nu\Delta\boldsymbol{u}'+\nabla p'=0 & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{u}'=0 & \text{in } \Omega,\\ \boldsymbol{u}'=0 & \text{on } \Gamma_1,\\ \boldsymbol{u}'=-\boldsymbol{h}_n\dfrac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} & \text{on } \Gamma_2, \end{cases} \tag{2.12}$$

where $\boldsymbol{h}_n=\boldsymbol{h}\cdot\boldsymbol{n}$ *is the normal component of the vector field* $\boldsymbol{h}$.

Proof. First, we establish continuous dependence of the solution $\boldsymbol{u}$ on variations of the boundary Γ_2. Then the differentiability of the solution $\boldsymbol{u}$ to the boundary value problem (2.1) with respect to the boundary Γ_2 is obtained as defined in (2.12).

To show the continuity, we consider the difference of $\tilde{\boldsymbol{u}}-\boldsymbol{u}$, and the variational equation yields

$$\begin{aligned} a(\tilde{\boldsymbol{u}}-\boldsymbol{u},\boldsymbol{v}) &= a(\tilde{\boldsymbol{u}},\boldsymbol{v})-a(B\tilde{\boldsymbol{u}},\boldsymbol{v})+a(B\tilde{\boldsymbol{u}},\boldsymbol{v})-a(B\tilde{\boldsymbol{u}},B\boldsymbol{v})+a(B\tilde{\boldsymbol{u}},B\boldsymbol{v})-\nu\int_\Omega\nabla(B\tilde{\boldsymbol{u}})\cdot\nabla(B\boldsymbol{v})A\,\mathrm{d}x\\ &\quad+\nu\int_\Omega\nabla(B\tilde{\boldsymbol{u}})\cdot\nabla(B\boldsymbol{v})A\,\mathrm{d}x-a(\boldsymbol{u},\boldsymbol{v})\\ &= a((I-B)\tilde{\boldsymbol{u}},\boldsymbol{v})+a(B\tilde{\boldsymbol{u}},(I-B)\boldsymbol{v})+\nu\int_\Omega\nabla(B\tilde{\boldsymbol{u}})\cdot\nabla(B\boldsymbol{v})(I-A)\,\mathrm{d}x+\int_\Omega(\tilde{\boldsymbol{f}}\cdot J_\varphi-\boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}x. \end{aligned}$$

According to the regularity and approximation (2.5)–(2.7), the perturbation argument shows the continuity

$$|\tilde{\boldsymbol{u}}_h-\boldsymbol{u}|_1\to 0 \quad \text{as } \|\boldsymbol{h}\|_{C^2(\Omega)}\to 0.$$

In order to show the differentiability, let $\boldsymbol{u}^*=\boldsymbol{u}'+\boldsymbol{h}\cdot\nabla\boldsymbol{u}$, and $\boldsymbol{u}^*$ be the solution of

$$a(\boldsymbol{u}^*,\boldsymbol{v})=\nu\int_\Omega\nabla\boldsymbol{u}(J_h+J_h^{\mathrm{T}}-\operatorname{div}\boldsymbol{h}\cdot I)\cdot\nabla\boldsymbol{v}\,\mathrm{d}x+\int_\Omega(\operatorname{div}\boldsymbol{h}\cdot\boldsymbol{f}+\boldsymbol{h}\cdot\nabla\boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}x \tag{2.13}$$

for all $\boldsymbol{v}\in\mathbf{V}(\Omega)$. Considering $\boldsymbol{u}^*$ is the solution of (2.13), we derive

$$\begin{aligned} a(\tilde{\boldsymbol{u}}-\boldsymbol{u}-\boldsymbol{u}^*,\boldsymbol{v}) &= a(\tilde{\boldsymbol{u}}-\boldsymbol{u},\boldsymbol{v})-a(\boldsymbol{u}^*,\boldsymbol{v})\\ &= \nu\int_\Omega\nabla((I-B)\tilde{\boldsymbol{u}})\cdot\nabla\boldsymbol{v}\,\mathrm{d}x+\nu\int_\Omega\nabla(B\tilde{\boldsymbol{u}})\cdot\nabla((I-B)\boldsymbol{v})\,\mathrm{d}x\\ &\quad+\nu\int_\Omega\nabla(B\tilde{\boldsymbol{u}})\cdot\nabla(B\boldsymbol{v})(I-A)\,\mathrm{d}x-\nu\int_\Omega\nabla\boldsymbol{u}(J_h+J_h^{\mathrm{T}}-\operatorname{div}\boldsymbol{h}\cdot I)\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\\ &\quad+\int_\Omega(\tilde{\boldsymbol{f}}\cdot J_\varphi-\boldsymbol{f}-\operatorname{div}\boldsymbol{h}\cdot\boldsymbol{f}-\boldsymbol{h}\cdot\nabla\boldsymbol{f})\cdot\boldsymbol{v}\,\mathrm{d}x. \end{aligned}$$

Let $\boldsymbol{v}=\tilde{\boldsymbol{u}}-\boldsymbol{u}-\boldsymbol{u}^*$, and apply the norm estimate (2.5)–(2.7) again:

$$\frac{1}{\|\boldsymbol{h}\|_{C^1}}|\tilde{\boldsymbol{u}}-\boldsymbol{u}-\boldsymbol{u}^*|_1\to 0 \quad \text{as } \|\boldsymbol{h}\|_{C^1}\to 0.$$

We split $\boldsymbol{u}^*$ into $\boldsymbol{h}\cdot\nabla\boldsymbol{u}$ and $\boldsymbol{u}'$. Taking use of Lemma 2.1 and the divergence formula, we obtain

$$\begin{aligned}
&a(\boldsymbol{h}\cdot\nabla\boldsymbol{u},\boldsymbol{v})\\
&\quad=-\nu\int_\Omega(\boldsymbol{h}\cdot\nabla\boldsymbol{u})\Delta\boldsymbol{v}\,\mathrm{d}x+\nu\int_{\partial\Omega}(\boldsymbol{h}\cdot\nabla\boldsymbol{u})\nabla\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s\\
&\quad=\nu\int_\Omega\nabla\boldsymbol{u}(J_h+J_h^{\mathrm{T}}-\operatorname{div}\boldsymbol{h}\cdot I)\cdot\nabla\boldsymbol{v}\,\mathrm{d}x-\nu\int_\Omega\operatorname{div}[\boldsymbol{b}]\,\mathrm{d}x\\
&\qquad+\nu\int_\Omega(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\Delta\boldsymbol{u}\,\mathrm{d}x+\nu\int_{\partial\Omega}(\boldsymbol{h}\cdot\nabla\boldsymbol{u})\nabla\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s\\
&\quad=\nu\int_\Omega\nabla\boldsymbol{u}(J_h+J_h^{\mathrm{T}}-\operatorname{div}\boldsymbol{h}\cdot I)\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\\
&\qquad-\nu\int_{\partial\Omega}((\boldsymbol{h}\cdot\nabla\boldsymbol{u})\nabla\boldsymbol{v}+(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\nabla\boldsymbol{u}-(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v})\boldsymbol{h})\cdot\boldsymbol{n}\,\mathrm{d}s\\
&\qquad-\int_\Omega(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\boldsymbol{f}\,\mathrm{d}x+\nu\int_{\partial\Omega}(\boldsymbol{h}\cdot\nabla\boldsymbol{u})\nabla\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s.
\end{aligned}$$

Therefore, the following identity holds:

$$\begin{aligned}
&a(\boldsymbol{h}\cdot\nabla\boldsymbol{u},\boldsymbol{v})\\
&\quad=a(\boldsymbol{u}^*,\boldsymbol{v})-\int_\Omega(\operatorname{div}\boldsymbol{h}\cdot\boldsymbol{f}+\boldsymbol{h}\cdot\nabla\boldsymbol{f})\boldsymbol{v}\,\mathrm{d}x\\
&\qquad-\nu\int_{\partial\Omega}((\boldsymbol{h}\cdot\nabla\boldsymbol{v})\nabla\boldsymbol{u}-(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v})\boldsymbol{h})\cdot\boldsymbol{n}\,\mathrm{d}s-\int_\Omega(\boldsymbol{h}\cdot\nabla\boldsymbol{v})\boldsymbol{f}\,\mathrm{d}x\\
&\quad=a(\boldsymbol{u}^*,\boldsymbol{v})-\int_\Omega\operatorname{div}(\boldsymbol{f}\boldsymbol{h}\boldsymbol{v})\,\mathrm{d}x-\nu\int_{\partial\Omega}((\boldsymbol{h}\cdot\nabla\boldsymbol{v})\nabla\boldsymbol{u}-(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v})\boldsymbol{h})\cdot\boldsymbol{n}\,\mathrm{d}s\\
&\quad=a(\boldsymbol{u}^*,\boldsymbol{v})-\int_{\partial\Omega}\boldsymbol{f}\boldsymbol{h}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s-\nu\int_{\partial\Omega}((\boldsymbol{h}\cdot\nabla\boldsymbol{v})\nabla\boldsymbol{u}-(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v})\boldsymbol{h})\cdot\boldsymbol{n}\,\mathrm{d}s.
\end{aligned}$$

Applying the decomposition formula in Lemma 2.2 and the boundary condition $\boldsymbol{v}|_{\partial\Omega}=0$, the equation yields

$$a(\boldsymbol{h}\cdot\nabla\boldsymbol{u},\boldsymbol{v})=a(\boldsymbol{u}^*,\boldsymbol{v}).\tag{2.14}$$

Thus, we obtain

$$a(\boldsymbol{u}',\boldsymbol{v})=0.$$

It is known that $\boldsymbol{u}|_{\Gamma_2}=0$ implies $\nabla_\tau\boldsymbol{u}|_{\Gamma_2}=0$. Note that $\boldsymbol{u}^*$ vanishes on the boundary Γ_2:

$$\boldsymbol{u}'=\boldsymbol{u}^*-\boldsymbol{h}\cdot\nabla\boldsymbol{u}=-\left(\boldsymbol{h}_\tau\cdot\nabla_\tau\boldsymbol{u}+\boldsymbol{h}\cdot\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\boldsymbol{n}\right)=-\boldsymbol{h}_n\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}.$$

Thus, $\boldsymbol{u}'$ satisfies the boundary value problem (2.12). This completes the proof. □

3. Numerical examples

This section describes the essential step of an algorithm for the recovering shape design of the Stokes problem, which we formulate in two dimension.

From the numerous methods that have been developed for the solution of inverse boundary value problems of this type, we note two groups of approaches, namely regularized Newton iterations and decomposition methods.

Newton method is based on the observation that the solution to problem (2.1) defines an operator F on set X of admissible boundaries by

$$F(\Gamma_2) = \boldsymbol{u}|_L, \tag{3.1}$$

where $\boldsymbol{u}$ is the solution of (2.1), and L is the line for measuring data. For simplicity, we can choose the measurement line L as a beeline in Ω, and also we can set it as a circle or an elliptic curve. $X := \{\varphi \in C^2(\Gamma_2), 0 < \beta \leqslant \|\varphi\|_{C^2} \leqslant \gamma\}$, and φ is the parametrized form of Γ_2.

However, since the linearized version of (3.1) inherits the ill-posedness, the Newton iterations need to be regularized. This approach has the advantages that, in principle, it is conceptually simple and that it leads to highly accurate reconstructions. But, as disadvantages, we note that the numerical implementation requires the forward solution of problem (2.1) in each step of the Newton iteration and reasonable a priori information for the initial approximation.

In any case the unknown boundary Γ_2 has to be parametrized. The boundary Γ_2 in polar coordinates is given by

$$\boldsymbol{x}_\alpha(t) = r_\alpha \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad 0 \leqslant t \leqslant 2\pi, \tag{3.2}$$

where

$$r_\alpha(t) = \alpha_0 + \sum_{j=1}^{M} [\alpha_j \cos jt + \alpha_{j+M} \sin jt] \tag{3.3}$$

with $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{2M})^{\mathrm{T}} \in R^{2M+1}$ for some fixed number $M \in \mathbf{N}$ (see [8]).

Let

$$U_M := \{\alpha \in R^{2M+1} : \rho_1 \leqslant r_\alpha(t) \leqslant \rho_2, t \in [0, 2\pi]\} \text{ for some } 0 < \rho_1 < \rho_2.$$

We can assign to each $\alpha \in U_M$ the cost function $F(\Gamma_2)(x_i), i = 1, \ldots, Q$.

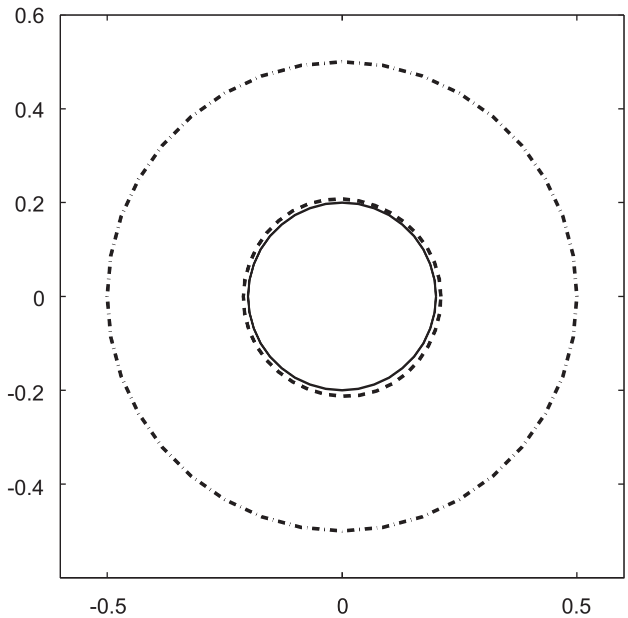

Fig. 1. Initial circle 0.5, Reynolds number = 10, relative error 1.29%, cpu time 338 s.

Theorem 3.1. *For $\alpha \in U_M$ the mapping F is differentiable with $\partial F_i(\alpha)/\partial\alpha_j = \partial_n \boldsymbol{u}'_j(x_i)$ for $i = 1, \ldots, Q$ and $j = 0, \ldots, 2M$. Here, $\boldsymbol{u}'_j \in V(\Omega)$ is the solution of the boundary value problem*

$$\begin{cases} -\nu\Delta\boldsymbol{u}'_j + \nabla p'_j = 0 & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}' = 0 & \text{in } \Omega, \\ \boldsymbol{u}'_j = 0 & \text{on } \Gamma_1, \\ \boldsymbol{u}'_j = -k\dfrac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{n}_i} & \text{on } \Gamma_2, \end{cases} \tag{3.4}$$

where

$$k = -\frac{r_\alpha(t)}{\sqrt{r'_\alpha(t)^2 + r_\alpha(t)^2}} \begin{cases} \cos jt, & j = 0, \ldots, M \\ \sin(j-M)t & j = M+1, \ldots, 2M \end{cases}$$

for $t \in [0, 2\pi)$.

The iteration algorithm consists of the following parts:

Step (1): Given an original boundary, Γ_2 can be described by the function α^0;

Step (2): Solve the direct problem (2.1) by the finite element method;

Step (3): For $j = 0, \ldots, 2M$, solve the boundary value problem (3.4);

Step (4): Give a suitable regularization parameter μ, and use the Newton method to obtain the new approximation;

Step (5): If the new approximation satisfies the stopping criterion, then terminate, otherwise go back to step (2).

We give some numerical examples in the case. In our numerical examples, the exterior boundary Γ_1 and interior boundary Γ_2 are given by the circles of radii 1 and 0.2, respectively, and center at the origin. We use the finite element method to simulate numerically. In Figs. 1–5, the full line for the interior curve represents the exact boundary, the dashed line gives the approximate boundary, and the dash-dot line denotes the initial curve. The cpu time is generated by performing on a home PC.

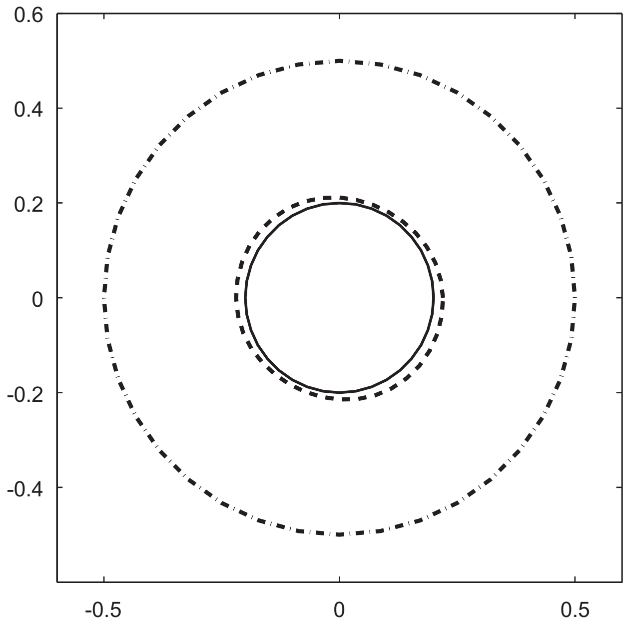

Fig. 2. Initial circle 0.5, Reynolds number = 200, relative error 3.77%, cpu time 572 s.

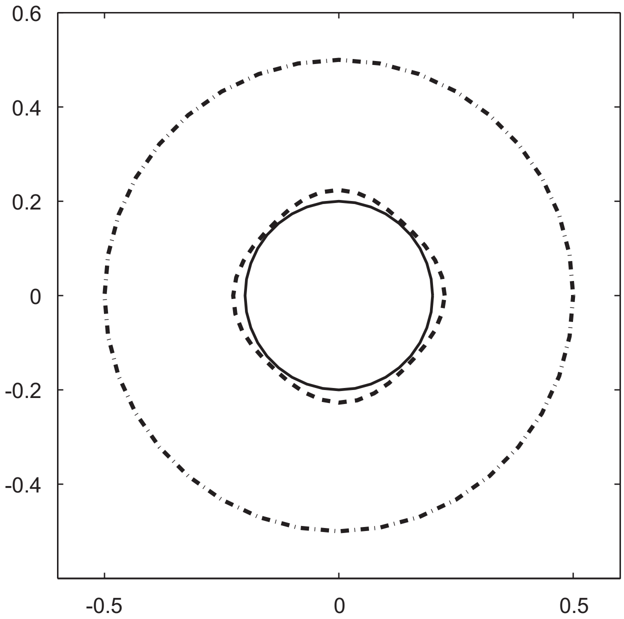

Fig. 3. Initial circle 0.5, Reynolds number = 500, relative error 6.13%, cpu time 987 s.

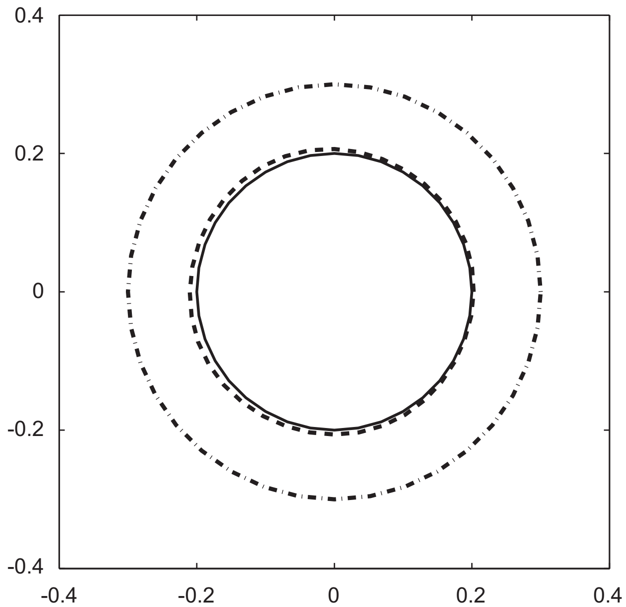

Fig. 4. Initial circle 0.3, Reynolds number = 100, no noise, relative error 2.08%, cpu time 431 s.

Figs. 1–3 display the results in which we choose the same initial circle (radius 0.5) but different Reynolds numbers (Reynolds number is the inverse of the coefficient of kinematic viscosity, i.e., $Re = 1/v$). The regularization parameter μ is 0.001.

Fig. 5 shows the reconstruction with 3% noise in the data but the same regularization parameters ($\mu = 0.005$) and the same initial circles (radius 0.3) as Fig. 4.

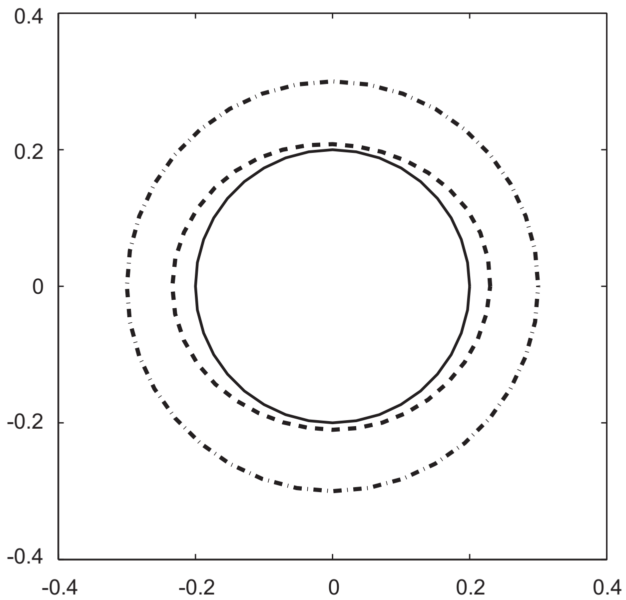

Fig. 5. Initial circle 0.3, Reynolds number = 100, 3% noise in the data, relative error 9.74%, cpu time 506 s.

4. Conclusions

The conclusions of the theoretical studies are as follows: we prove the differentiability of the solution to the boundary value problem, and derive the representation for domain derivative of the corresponding operator. This allows the investigation of the regularized iterative method for the solution of this ill-posed and nonlinear problem. The results of several numerical experiments show that the iterative algorithm gives good reconstruction, and indicate the feasibility of the regularized Newton method. Further research is necessary on efficient implementations for time-dependent Navier–Stokes flow and real problems in the industry.

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